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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. It should be remembered that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with more familiar results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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PROGRESS REPORT.

It is with great sorrow that the Committee records the sad death, in November last, of Mr. B. Hanumantha Rao, B.A., I.S.O., who was the first President of the Society. It was under his fostering and sympathetic guidance that the Society, in its earlier stages, developed into a prosperous and useful institution.

The Committee also records with regret the premature death, on the 13th November 1922, of Mr. C. Krishnamachary, who was an enthusiastic member of the Society, and a frequent contributor to this Journal.

The following gentlemen have been admitted as Members of the Society:—

(1) K. N. Sundaresan, B.A. (Hons.), Lecturer in Mathematics, Kallikota College, Berhampore.

(2) T. Totadri Aiyangar, B.A. (Hons.), University Research Student, $\frac{8}{10}$ A, Kolandavelu Street, Purasawalkum, Madras.—(at concessional rate).

(3) S. Venkatasubba Rao, B.A., L.T., Second Assistant, Board High School, Arni.—(at concessional rate).

P. V. SESHU AIYAR,
Hon. Joint Secretary.

GENERAL PERSYMMETRIC DETERMINANTS*

By C. Krishnamachari and M. Bhimasena Rao.

In our two previous papers on determinants, we evaluated some simple cases of persymmetrics, the first element being 1, A_1, or A_2. In this paper, we evaluate when the first element is A_r. Dr. Muir, on page 350 in Vol. II of his "Theory of Determinants," gives a single instance of the evaluation by Brioschi of a persymmetric in the general case. The authors are not aware of any other evaluations.

§ 1. We begin by noticing that the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ a_2 & a_3 & a_4 & \cdots & a_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & a_{n+1} & \cdots & \cdots & a_{2n} \end{vmatrix} \quad \cdots \quad (1)$$

can be easily transformed into the persymmetric

$$[\Delta a_1,\ \Delta^n a_1]$$

where

$$\Delta a_r = a_{r+1} - a_r.$$

It is known that

$$[a_1, a_n] = [\Delta^0 a_1,\ \Delta^{n-1} a_1] \quad \text{(Scott : \textit{Determinants}, p. 100.)}$$

Similarly

$$\begin{vmatrix} 1 & -1 & 1 & -1 & \cdots & (-1)^n \\ a_1 & a_2 & a_3 & \cdots & \cdots & a_{n+1} \\ a_2 & a_3 & a_4 & \cdots & \cdots & a_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & \cdots & a_{2n} \end{vmatrix} = [\mathrm{E}a_1,\ \mathrm{E}^n a_1], \quad \cdots \quad (2)$$

where

$$\mathrm{E}a_r = a_r + a_{r+1}.$$

§ 2. We have from the equations (14·2) and (14·3) of our last paper, Vol. XIV, page 131,

$$\begin{aligned} A_{2m+1} - {}_{2m}\alpha_{2m} A_{2m} + {}_{2m}\alpha_{2m-2} A_{2m-1} - \cdots & \\ + (-1)^m\, {}_{2m}\alpha_2\, A_{m+1} &= a_1 a_2 \ldots\ldots a_{2m+1} \\ A_{2m} - {}_{2m}\alpha_{2m} A_{2m-1} + \cdots + (-1)^m\, {}_{2m}\alpha_2\, A_m &= 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \quad \cdots \\ A_{m+2} - {}_{2m}\alpha_{2m} A_{m+1} + \cdots + (-1)^m\, {}_{2m}\alpha_2\, A_2 &= 0 \end{aligned}$$

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* [*Continued from page 138, Vol. XIV, No. 4, J. I. M. S.*]

From these m equations, we can solve for the m quantities ${}_{2m}\alpha_{2r}$. Solving for ${}_{2m}\alpha_2$, we have

$$[A_2, A_{m+1}]\ {}_{2m}\alpha_2 = \begin{vmatrix} A_3 & A_4 & \dots & A_{m+2} \\ A_4 & A_5 & \dots & A_{m+3} \\ \dots & \dots & \dots & \dots \\ A_{m+2} & A_{m+3} & \dots & A_{2m+1} - a_1 a_2 \dots a_{2m+1} \end{vmatrix}$$

$$= [A_3, A_{m+2}] - a_1 a_2 \dots a_{2m+1} [A_3, A_{m+1}].$$

Now

$$[A_2, A_{m+1}] = {}_{2m}\alpha_2 \,.\, a_1^m (a_2 a_3)^{m-1} \dots (a_{2m-2}\ a_{2m-1})^1$$

$$\therefore\quad [A_3, A_{m+2}] = a_1 a_2 \dots a_{2m+1} [A_3, A_{m+1}]$$
$$+ ({}_{2m}\alpha_2)^2 a_1^m (a_2 a_3)^{m-1} \dots (a_{2m-2}\, a_{2m-1})^1$$

$$\therefore\quad \frac{[A_3,\ A_{m+2}]}{a_1^{m+1} (a_2 a_3)^m \dots (a_{2m}\ a_{2m+1})^1}$$

$$= \frac{({}_{2m}\alpha_2)^2}{a_1 a_2 \dots a_{2m+1}} + \frac{[A_3,\ A_{m+1}]}{a_1^m (a_2 a_3)^{m-1} \dots (a_{2m-2}\ a_{2m-1})^1}$$

$$= \frac{({}_{2m}\alpha_2)^2}{a_1 a_2 \dots a_{2m+1}} + \frac{({}_{2m-2}\alpha_2)^2}{a_1 a_2 \dots a_{2m-1}} + \dots + \frac{({}_2\alpha_2)^2}{a_1 a_2 a_3} + \frac{1}{a_1}. \quad \dots (3)$$

Thus we obtain the value of the persymmetric $[A_3, A_{m+2}]$.

§ 3. It is easily seen by induction that if $\frac{p_n}{q_n}$ is the nth convergent of the continued fraction

$$\frac{1}{x -}\ \frac{a_1}{x -}\ \frac{a_2}{x -}\ \frac{a_3}{x -}\ \dots\dots,$$

then

$$q_{2n} = x^{2n} - {}_{2n-1}\alpha_{2n-1}\, x^{2n-2} + {}_{2n-1}\alpha_{2n-3}\, x^{2n-4} - \dots\dots$$
$$+ (-1)^n\, {}_{2n-1}\alpha_1 \quad \dots \quad \dots \quad (4)$$

$$q_{2n+1} = x^{2n+1} - {}_{2n}\alpha_{2n}\, x^{2n-1} + {}_{2n}\alpha_{2n-2}\, x^{2n-3} - \dots\dots$$
$$+ (-1)^n\, {}_{2n}\alpha_2\, x. \quad \dots \quad \dots \quad (5)$$

Combining (4) with the equations (13·3) of the last paper (Vol. XIV, page 130), we obtain

$$\begin{vmatrix} 1 & x^2 & x^4 & \dots & x^{2n} \\ A_0 & A_1 & A_2 & \dots & A_n \\ A_1 & A_2 & A_3 & \dots & A_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ A_{n-1} & A_n & \dots & \dots & A_{2n-1} \end{vmatrix} = (-1)^n q_{2n} [A_0, A_{n-1}]. \quad \dots \quad (6)$$

Similarly from (5) and (14·3) of the same paper, (Vol. XIV, page 131), we have

$$\begin{vmatrix} x & x^3 & x^5 & \cdots & x^{2n+1} \\ A_1 & A_2 & \cdots & \cdots & A_{n+1} \\ A_2 & A_3 & \cdots & \cdots & A_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n+1} & \cdots & \cdots & A_{2n} \end{vmatrix} = (-1)^n q_{2n+1} [A_1, A_n]. \quad \ldots \quad (7)$$

From (6) and (7), by equating the co-efficients of the various powers of x, we obtain the values of determinants obtained by omitting one column of a persymmetric.

Ex. As examples of (6) and (7), we may obtain Rouche's expressions for Legendre's polynomials. (See Muir: *Theory of Determinants*, Vol. II, page 356).

Consider the continued fraction for $\tanh^{-1}\left(\frac{1}{x}\right)$, *viz.*,

$$\frac{1}{x-}\,\frac{1^2}{3x-}\,\frac{2^2}{5x-}\,\frac{3^2}{7x-}\cdots\cdots\cdots$$

The relation between the denominators of the successive convergents is

$$\left.\begin{aligned} q_{n+1} &= (2n+1)\, x\, q_n - n^2\, q_{n-1} \\ \text{with the initial values}\quad q_1 &= x, \quad q_2 = 3x^2 - 1. \end{aligned}\right\} \qquad (8)$$

Now if we write $P_n' = n!\, P_n$ where P_n is Legendre's polynomial of the n^{th} degree, the well known relation

$$(n+1)\, P_{n+1}(z) - (2n+1)\, z\, P_n(z) + n\, P_{n-1}(z) = 0$$

$$\left.\begin{aligned} \text{becomes}\quad P'_{n+1}(z) &= (2n+1)\, z\, P'_n(z) - n^2\, P'_{n-1}(z) \\ \text{and}\quad P_1'(x) &= x, \quad P_2'(x) = 3x^2 - 1. \end{aligned}\right\} \qquad (9)$$

From (8) and (9), it follows that

$$q_{n+1}(x) = (n+1)!\; P_{n+1}(x). \qquad (10)$$

Hence from (6), (7) we have

$$\begin{vmatrix} 1 & x^2 & x^4 & \cdots & x^{2n} \\ 1 & \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{2n+1} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2n-1} & \frac{1}{2n+1} & \cdots & \cdots & \frac{1}{4n-1} \end{vmatrix} = (-1)^n\, (2n)!\; P_{2n}(x). \left[1, \frac{1}{2n-1}\right] \quad \ldots \quad (11)$$

and

$$\begin{vmatrix} x & x^3 & \dots & x^{2n+1} \\ \frac{1}{3} & \frac{1}{5} & \dots & \frac{1}{2n+3} \\ \dots & \dots & \dots & \dots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \dots & \frac{1}{4n+1} \end{vmatrix} = (-1)^n\,(2n+1)\,!\;P_{2n+1}(x). \qquad \left[\tfrac{1}{3},\, \frac{1}{2n+1}\right] \quad \dots(12)$$

The values of the persymmetrics $\left[1, \dfrac{1}{2n-1}\right]$ and $\left[\dfrac{1}{3}, \dfrac{1}{2n+1}\right]$ are given by us in our last paper (Vol. XIV, page 135). It does not appear that these have been evaluated before. Rouche seems to state only that the above determinants are Legendre's polynomials multiplied by constants.

§ 4. Now eliminate x between q_{2n+1} and q_{2n-1}. The result is

$$x^{2n+1}\begin{vmatrix} 1 & {}_{2n}\alpha_2 \\ 0 & {}_{2n-2}\alpha_2 \end{vmatrix} - x^{2n-1}\begin{vmatrix} {}_{2n}\alpha_{2n} & {}_{2n}\alpha_2 \\ 1 & {}_{2n-2}\alpha_2 \end{vmatrix} + x^{2n-3}\begin{vmatrix} {}_{2n}\alpha_{2n-2} & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_{2n-2} & {}_{2n-2}\alpha_2 \end{vmatrix} \dots$$
$$+(-1)^{n-1}\,x^3\begin{vmatrix} {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \end{vmatrix} = \begin{vmatrix} q_{2n+1} & {}_{2n}\alpha_2 \\ -q_{2n-1} & {}_{2n-2}\alpha_2 \end{vmatrix} \qquad (13)$$

Similarly in equations (14·3). *Ibid* page 131, take the general equation

$$A_{n+r} - {}_{2n}\alpha_{2n}\,A_{n+r-1} + {}_{2n}\alpha_{2n-2}\,A_{n+r-2} - \dots + (-1)^n\,{}_{2n}\alpha_2\,A_r = 0.$$

Write $(n-1)$ for n, and eliminate A_r between the two equations. The result is

$$A_{n+r}\begin{vmatrix} 1 & {}_{2n}\alpha_2 \\ 0 & {}_{2n-2}\alpha_2 \end{vmatrix} - A_{n+r-1}\begin{vmatrix} {}_{2n}\alpha_{2n} & {}_{2n}\alpha_2 \\ 1 & {}_{2n-2}\alpha_2 \end{vmatrix} + A_{n+r-2}\begin{vmatrix} {}_{2n}\alpha_{2n-2} & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_{2n-2} & {}_{2n-2}\alpha_2 \end{vmatrix}$$
$$- \dots + (-1)^{n-1}\,A_{r+1}\begin{vmatrix} {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \end{vmatrix} = 0. \qquad (14)$$

We have $(n-1)$ equations (14), *viz.*, for values of r from 1 to $n-1$.

In a similar manner, we have from the first of equations (14·3) and that obtained by writing $(n-1)$ for n in (14·2),

$$A_{2n}\begin{vmatrix} 1 & {}_{2n}\alpha_2 \\ 0 & {}_{2n-2}\alpha_2 \end{vmatrix} - A_{2n-1}\begin{vmatrix} {}_{2n}\alpha_{2n} & {}_{2n}\alpha_2 \\ 1 & {}_{2n-2}\alpha_2 \end{vmatrix} + \dots \quad \dots \quad \dots$$
$$+ (-1)^{n-1}\,A_{n+1}\begin{vmatrix} {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \end{vmatrix} = (a_1 a_2 \dots a_{2n-1})\;{}_{2n}\alpha_2. \qquad (15)$$

From the $(n+1)$ equations (13), (14) and (15), eliminate the n co-efficients. We obtain

$$\begin{vmatrix} {}_{2n}\alpha_2\, q_{2n-1} + {}_{2n-2}\alpha_2\, q_{2n+1} & x^3 & x^5 & \dots & x^{2n+1} \\ a_1 a_2 \dots a_{2n+1}\ {}_{2n}\alpha_2 & A_{n+1} & A_{n+2} & \dots & A_{2n} \\ 0 & A_n & A_{n+1} & \dots & A_{2n-1} \\ 0 & \dots & \dots & \dots & \dots \\ \dots & A_2 & A_3 & \dots & A_{n+1} \end{vmatrix} = 0.$$

i.e.,

$$a_1 a_2 \dots a_{2n-1}\ {}_{2n}\alpha_2 \begin{vmatrix} x^3 & x^5 & \dots & x^{2n+1} \\ A_2 & A_3 & \dots & A_{n+1} \\ A_3 & A_4 & \dots & A_{n+2} \\ \dots & \dots & \dots & \dots \\ A_n & A_{n+1} & \dots & A_{2n-1} \end{vmatrix}$$

$$= (-1)^{n-1} \begin{vmatrix} q_{2n+1} & {}_{2n}\alpha_2 \\ -q_{2n-1} & {}_{2n-2}\alpha_2 \end{vmatrix} [A_2, A_{n+1}]$$

i.e.,

$$\begin{vmatrix} x^3 & x^5 & \dots & x^{2n+1} \\ A_2 & A_3 & \dots & A_{n+1} \\ \dots & \dots & \dots & \dots \\ A_n & A_{n+1} & \dots & A_{2n-1} \end{vmatrix}$$

$$= (-1)^{n-1}\ [A_1, A_{n-1}] \begin{vmatrix} q_{2n+1} & {}_{2n}\alpha_2 \\ -q_{2n-1} & {}_{2n-2}\alpha_2 \end{vmatrix} \quad \dots \quad (16)$$

from which again, we can obtain the values of determinants by equating co-efficients. In particular, equating the co-efficients of x^3, we have

$$[A_3\ A_{n+1}] = \begin{vmatrix} {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \end{vmatrix} \times$$

$$a_1^{\,n-1} (a_2 a_3)^{n-2} \ \dots\ (a_{2n-4}\ a_{2n-3})^{*}. \qquad \dots \quad (17)$$

---

* Comparing (17) with (3) above, we have

$$\begin{vmatrix} {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \end{vmatrix}$$

$$= a_1 a_2 \dots a_{2n-1} \left\{ \frac{1}{a_1} + \frac{({}_2\alpha_2)^2}{a_1 a_2 a_3} + \dots + \frac{({}_{2n-2}\alpha_2)^2}{a_1 a_2 a_3 \dots a_{2n-1}} \right\}.$$

It is easy to expand the left-hand side into the series on the right.

For

$${}_{2n}\alpha_4\ {}_{2n-2}\alpha_2 - {}_{2n-2}\alpha_4 \cdot {}_{2n}\alpha_2 = {}_{2n-1}\alpha_3 \cdot {}_{2n-2}\alpha_2 - {}_{2n-1}\alpha_1\ {}_{2n-2}\alpha_4$$

and

$$\frac{1}{a_1 a_2 \dots a_{2n-1}} ({}_{2n-1}\alpha_3 \cdot {}_{2n-2}\alpha_2 - {}_{2n-1}\alpha_1 \cdot {}_{2n-2}\alpha_4)$$

$$= \frac{({}_{2n-2}\alpha_2)^2}{a_1 a_2 \dots a_{2n-1}} + \frac{1}{a_1 a_2 \dots a_{2n-3}} \{ {}_{2n-3}\alpha_3 \cdot {}_{3n-4}\alpha_2 - {}_{2n-3}\alpha_1 \cdot {}_{2n-4}\alpha_4 \}$$

from which the series on the right follows.

The equation (16) can also be expressed in slightly different forms obtained by other but similar methods of procedure.

Again take the equation (13), obtain an equation (13′) by writing $(n-1)$ for n and between the two eliminate x^3, and obtain an equation (13″). The expression on the right of (13″) is easily seen to be

$$L = {}_{2n-2}\alpha_2 \begin{vmatrix} q_{2n+1} & {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ -q_{2n-1} & {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \\ q_{2n-3} & {}_{2n-4}\alpha_4 & {}_{2n-4}\alpha_2 \end{vmatrix}$$

Similarly obtain (15′) by writing $(n-1)$ for n in (15), and the last of the equations $(14, r = n-1)$, eliminate A_n. The expression on the right in (15″) thus obtained is

$$M = a_1 a_2 \ldots a_{2n-3}\, {}_{2n-2}\alpha_2 \begin{vmatrix} {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \end{vmatrix}$$

From the remaining $(n-2)$ equations (14), obtain $(n-2)$ equations (14″) in exactly the same way as (13″) was obtained from (13), so that we have in all n equations with double dashes. It is obvious that the co-efficients in all these equations are the same, and are $(n-1)$ in number. (We have co-efficients for odd powers of x from x^5 to x^{2n+1}). Eliminating the co-efficients, after simplification,

$$\begin{vmatrix} q_{2n+1} & {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ -q_{2n-1} & {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \\ q_{2n-3} & {}_{2n-4}\alpha_4 & {}_{2n-4}\alpha_2 \end{vmatrix} [A_3,\ A_{n+1}] = (-1)^{n-2}\, a_1 a_2 \ldots a_{2n-3} \times$$

$$\begin{vmatrix} {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \end{vmatrix} \begin{vmatrix} x^5 & x^7 & \ldots & x^{2n+1} \\ A_3 & A_4 & \ldots & A_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ A_n & A_{n+1} & \ldots & A_{2n-2} \end{vmatrix}$$

Substituting for $[A_3,\ A_{n+1}]$ its value from (17), we obtain finally

$$\begin{vmatrix} x^5 & x^7 & \ldots & x^{2n+1} \\ A_3 & A_4 & \ldots & A_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ A_n & A_{n+1} & \ldots & A_{2n-2} \end{vmatrix} = (-1)^{n-2} \begin{vmatrix} q_{2n+1} & {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ -q_{2n-1} & {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \\ q_{2n-3} & {}_{2n-4}\alpha_4 & {}_{2n-4}\alpha_2 \end{vmatrix} \times$$

$$a_1^{\,n-2} (a_2 a_3)^{n-3} \ldots (a_{2n-6}\, a_{2n-5})^1. \qquad \ldots \quad (18)$$

$$= (-1)^{n-2} \begin{vmatrix} x^2 q_{2n-1} & {}_{2n-2}\alpha_2 & 0 \\ -q_{2n-1} & {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \\ q_{2n-3} & {}_{2n-4}\alpha_4 & {}_{2n-4}\alpha_2 \end{vmatrix} \times$$

$$a_1^{\,n-2} (a_2 a_3)^{n-3} \ldots (a_{2n-6}\, a_{2n-5})^1.$$

Equating the co-efficients of x^5, we have

$$[A_4,\ A_{n+1}] = \begin{vmatrix} {}_{2n}\alpha_6 & {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_6 & {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 \\ {}_{2n-4}\alpha_6 & {}_{2n-4}\alpha_4 & {}_{2n-4}\alpha_2 \end{vmatrix} \times$$

$$a_1^{\,n-2}\,(a_2a_3)^{n-3}\ \dots\ (a_{2n-6}a_{2n-5})^1$$

$$= \begin{vmatrix} {}_{2n-2}\alpha_4 & {}_{2n-2}\alpha_2 & 0 \\ {}_{2n-2}\alpha_6 & {}_{2n-2}\alpha_2 & {}_{2n-2}\alpha_2 \\ {}_{2n-4}\alpha_6 & {}_{2n-4}\alpha_4 & {}_{2n-4}\alpha_2 \end{vmatrix} \times$$

$$a_1^{\,n-2}(a_2a_3)^{n-3}\ \dots\ (a_{2n-6}\ a_{2n-5})^1. \qquad \dots \quad (19)$$

The equations (18) and (19) suggest the following generalization :—

$$\begin{vmatrix} x^{2r+1} & x^{2r+3} & \dots & x^{2n+1} \\ A_{r+1} & A_{r+2} & \dots & A_{n+1} \\ \dots & \dots & \dots & \dots \\ A_n & A_{n+r} & \dots & A_{n-r} \end{vmatrix}$$

$$=(-1)^{n-r} \begin{vmatrix} q_{2n+1} & {}_{2n}\alpha_{2r} & {}_{2n}\alpha_{2r-2} & \dots & {}_{2n}\alpha_2 \\ -q_{2n-1} & {}_{2n-2}\alpha_{2r} & {}_{2n-2}\alpha_{2r-2} & \dots & {}_{2n-2}\alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ (-1)^r\, q_{2n-2r+1} & {}_{2n-2r+2}\alpha_{2r} & & \dots & {}_{2n-2r+2}\alpha_2 \end{vmatrix} \times$$

$$a_1^{\,n-r}\,(a_2a_3)^{n-r-1}\dots\ (a_{2n-2r-2}\ a_{2n-2r-1})^1 \qquad \dots \quad (20)$$

and

$$[A_{r+2},\ A_{n+1}] = a_1^{\,n-r}(a_2a_3)^{n-r-1}\ \dots\ (a_{2n-2r-2}\ a_{2n-2r-1})^1 \times$$

$$\begin{vmatrix} {}_{2n}\alpha_{2r+2} & {}_{2n}\alpha_{2r} & \dots & {}_{2n}\alpha_2 \\ {}_{2n-2}\alpha_{2r+2} & {}_{2n-2}\alpha_{2r} & \dots & {}_{2n-2}\alpha_2 \\ \dots & \dots & \dots & \dots \\ {}_{2n-2r}\alpha_{2r+2} & \dots & \dots & {}_{2n-2r}\alpha_2 \end{vmatrix} \qquad (21)$$

This gives also the general persymmetric involving Euler's and various other numbers which can be obtained by the table.

The equations (20) and (21) are always true provided we remember that whenever the left suffix of α is just 2 less than the right, its value is unity, and whenever the left suffix is more than 2 less, its value is zero. (21) also expresses an A as the value of a determinant. Thus writing $r = n - 1$, we get

$$A_{n+1} = a_1 \times \begin{vmatrix} {}_{2n}\alpha_{2n} & {}_{2n}\alpha_{2n-2} & \cdots & {}_{2n}\alpha_2 & \cdots & \cdots & \cdots & \cdots \\ 1 & {}_{2n-2}\alpha_{2n-2} & \cdots & {}_{2n-2}\alpha_2 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & {}_{2n-4}\alpha_2 & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots & {}_{2n-6}\alpha_2 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 & {}_4\alpha_4 & {}_4\alpha_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & {}_2\alpha_2 \end{vmatrix} \qquad (22)$$

there being n rows and columns. (22) may also be easily transformed into

$$a_1 \times \begin{vmatrix} {}_2\alpha_2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ {}_3\alpha_1 & {}_3\alpha_3 & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ {}_5\alpha_1 & {}_5\alpha_3 & {}_5\alpha_5 & 1 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ {}_{2n-3}\alpha_1 & {}_{2n-3}\alpha_3 & \cdots & \cdots & \cdots & \cdots & {}_{2n-3}\alpha_{2n-3} & 1 \\ {}_{2n-1}\alpha_1 & {}_{2n-1}\alpha_3 & \cdots & \cdots & \cdots & \cdots & \cdots & {}_{2n-1}\alpha_{2n-1} \end{vmatrix} \qquad (23)$$

In particular if $a_n = n^r$, we have E_n as the value of a determinant. *e.g.*

$$1385 = E_4 = \begin{vmatrix} 91 & 1519 & 3429 \\ 1 & 30 & 89 \\ 0 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 55 & 439 & 225 \\ 1 & 14 & 9 \\ 0 & 1 & 5 \end{vmatrix}$$

§ 4·1. The determinant in (21) on the right can be transformed in an interesting way which makes calculation simpler.

The equation (21) may be written as

$$[A_r, A_{n+r}] = a_1^{n+1} (a_2 a_3)^n \ldots (a_{2n}\, a_{2n+1})^1 \times$$

$$\begin{vmatrix} {}_{2n+2}\alpha_2 & {}_{2n+2}\alpha_4 & \cdots & {}_{2n+2}\alpha_{2r-2} \\ {}_{2n+4}\alpha_2 & {}_{2n+4}\alpha_4 & \cdots & {}_{2n+4}\alpha_{2r-2} \\ {}_{2n+6}\alpha_2 & {}_{2n+6}\alpha_4 & \cdots & {}_{2n+6}\alpha_{2r-2} \\ \cdots & \cdots & \cdots & \cdots \\ {}_{2n+2r-2}\alpha_2 & \cdots & \cdots & {}_{2n+2r-2}\alpha_{2r-2} \end{vmatrix} \qquad (21)$$

Transform the determinant on the right thus :—Retain the first row as it is ; for the new second row, subtract a_{2n+4} times the corresponding first row elements ; for the new third row, subtract a_{2n+6} times the corresponding elements of the present second row ; and so on.

The transformed determinant is

$$\begin{vmatrix} {}_{2n+2}a_2 & {}_{2n+2}a_4 & \cdots & {}_{2n+2}a_{2r-2} \\ {}_{2n+3}a_1 & {}_{2n+3}a_3 & \cdots & {}_{2n+3}a_{2r-3} \\ {}_{2n+5}a_1 & {}_{2n+5}a_3 & \cdots & {}_{2n+5}a_{2r-3} \\ \cdots & \cdots & \cdots & \cdots \\ {}_{2n+2r-3}a_1 & {}_{2n+2r-3}a_3 & \cdots & {}_{2n+2r-3}a_{2r-3} \end{vmatrix}$$

Transform this determinant also in a similar way. *i.e.*, retain the first two rows ; for the new third row, subtract a_{2n+5} times the second ; for the fourth row, a_{2n+7} times the third and so on. The new transformed determinant is

$$\begin{vmatrix} {}_{2n+2}a_2 & {}_{2n+2}a_4 & \cdots & {}_{2n+2}a_{2r-2} \\ {}_{2n+3}a_1 & {}_{2n+3}a_3 & \cdots & {}_{2n+3}a_{2r-3} \\ 0 & {}_{2n+4}a_2 & \cdots & {}_{2n+4}a_{2r-4} \\ \cdots & {}_{2n+6}a_2 & \cdots & {}_{2n+6}a_{2r-4} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & {}_{2n+2r-4}a_2 & \cdots & {}_{2n+2r-4}a_{2r-4} \end{vmatrix}$$

Again proceed in a similar manner with all except the first three rows, and so on. We finally arrive at the new determinant

$$\begin{vmatrix} {}_{2n+2}a_2 & {}_{2n+2}a_4 & \cdots & \cdots & {}_{2n+2}a_{2r-2} \\ {}_{2n+3}a_1 & {}_{2n+3}a_3 & \cdots & \cdots & {}_{2n+3}a_{2r-3} \\ 0 & {}_{2n+4}a_2 & \cdots & \cdots & {}_{2n+4}a_{2r-4} \\ 0 & {}_{2n+5}a_1 & \cdots & \cdots & {}_{2n+5}a_{2r-5} \\ \cdots & 0 & \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & {}_{2n+r}a_{r-2} & {}_{2n+r}a_r \end{vmatrix}$$

of $(r-1)$ rows and columns.

The corresponding equation (23) is

$$A_{n+1} = a_1 \times \begin{vmatrix} {}_2a_2 & 1 & 0 & 0 & \cdots & \cdots & \cdots \\ {}_3a_1 & {}_3a_3 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & {}_4a_2 & {}_4a_4 & 1 & 0 & \cdots & \cdots \\ 0 & {}_5a_1 & {}_5a_3 & {}_5a_5 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (23')$$

of n rows and columns, which can be directly obtained from the system of relations between the A's already established.

§ 4·2. The evaluation of the persymmetric expressed in (21) can be proved directly by a multiplication of determinants. The method is perfectly general; for the sake of clearness and simplicity, we content ourselves with proving that

$$[A_s,\ A_{n+2}] = a_1^{\,n}\ (a_2a_3)^{n-1}\ \dots\ (a_{2n-2}\ a_{2n-1})^1 \begin{vmatrix} {}_{2n+2}\alpha_4 & {}_{2n+2}\alpha_2 \\ {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \end{vmatrix}$$

$$= \begin{vmatrix} {}_{2n}\alpha_2 & {}_{2n}\alpha_4 \\ {}_{2n+2}\alpha_2 & {}_{2n+2}\alpha_4 \end{vmatrix} \begin{vmatrix} A_1 & A_2 & \dots & A_n \\ A_2 & A_3 & \dots & A_{n+1} \\ A_3 & A_4 & \dots & A_{n+2} \\ \dots & \dots & \dots & \dots \\ A_n & A_{n+1} & \dots & A_{2n-1} \end{vmatrix}$$

Converting the small determinant on the right into one of n rows and columns by introducing 1 along the diagonal line, we have

· Right-hand side expression

$$= -\begin{vmatrix} {}_{2n}\alpha_2 & -{}_{2n}\alpha_4 & 0 & 0 & \dots \\ {}_{2n+2}\alpha_2 & -{}_{2n+2}\alpha_4 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} A_1 & A_2 & \dots & A_n \\ A_2 & A_3 & \dots & A_{n+1} \\ A_3 & A_4 & \dots & A_{n+2} \\ \dots & \dots & \dots & \dots \\ A_n & A_{n+1} & \dots & A_{2n-1} \end{vmatrix}$$

$$= \left| \begin{array}{ccc} A_{1\,2n}\alpha_2 - A_{2\,2n}\alpha_4 & A_{2\ 2n}\alpha_2 - A_{3\ 2n}\alpha_4 & \dots \\ A_{1\,2n+2}\alpha_2 - A_{2\,2n+2}\alpha_4 & \dots & \dots \\ A_3 & A_4 & \dots \\ A_4 & A_5 & \dots \\ \dots & \dots & \dots \\ A_n & A_{n+1} & \dots \end{array} \right.$$

$$\left. \begin{array}{ccc} A_{r\ 2n}\alpha_2 - A_{r+1\,2n}\alpha_4 & \dots & A_{n\ 2n}\alpha_2 - A_{n+1\,2n}\alpha_4 \\ A_{r\,2n+2}\alpha_2 - A_{r+1\,2n+2}\alpha_4 & \dots & A_{n\,2n+2}\alpha_2 - A_{n+1\,2n+2}\alpha_4 \\ \dots & \dots & A_{n+2} \\ \dots & \dots & A_{n+3} \\ \dots & \dots & \dots \\ \dots & \dots & A_{2n-1} \end{array} \right|$$

by the ordinary rule for the multiplication of two determinants.

Now we have the relation

$$
{}_{2n}\alpha_2 A_r - {}_{2n}\alpha_4 A_{r+1} + {}_{2n}\alpha_6 A_{r+2} - \dots + (-1)^{n-1}\, {}_{2n}\alpha_{2n} A_{n+r-1} + (-1)^n A_{n+r} = 0
$$

valid for values of r from 1 to n. (See 14·3 *Ibid*).

$$
\therefore\quad A_{n+r} = (-1)^{n-1} \{ {}_{2n}\alpha_2 A_r - {}_{2n}\alpha_4 A_{r+1} + {}_{2n}\alpha_6 A_{r+2} - \dots + (-1)^{n-1}\, {}_{2n}\alpha_{2n} A_{n+r-1} \}. \qquad \dots \quad (24)
$$

Transform the last determinant by taking for the new first row that obtained by adding to it $(-1)^2\ {}_{2n}\alpha_6$. 3rd row, $(-1)^3\ {}_{2n}\alpha_8$. 4th row, ... , $(-1)^{n-1}\ {}_{2n}\alpha_{2n}$. nth row corresponding elements, and substituting for the result thus obtained from equation (24). The general term in the first row is

$$
A_r\, {}_{2n}\alpha_2 - A_{r+1}\, {}_{2n}\alpha_4 + A_{r+2}\, {}_{2n}\alpha_6 - \dots + (-1)^{n-1}\, {}_{2n}\alpha_{2n} A_{n+r-1} = (-1)^{n-1} A_{n+r}.
$$

Thus the new elements of the first row are

$$
A_{n+1},\ A_{n+2},\ \dots\ A_{2n}
$$

multiplied by $(-1)^{n-1}$. In a similar manner obtain the new second row from the old. The new second row elements are easily seen to be

$$
A_{n+2},\ A_{n+3},\ \dots\ A_{2n+1}
$$

multiplied by $(-1)^n$.

Rearranging the last after these substitutions, we see that the product is equal to $[A_3,\ A_{n+2}]$.

It is now sufficiently obvious that the proof is perfectly general, and thus we establish (21).

§ 4·3. Taking a suitable system of equations and reducing, we can prove that

$$
\begin{vmatrix} x^7 & x^9 & \dots & x^{2n+5} \\ A_4 & A_5 & \dots & A_{n+3} \\ \dots & \dots & \dots & \dots \\ A_{n+2} & \dots & \dots & A_{2n+1} \end{vmatrix} - a_1 a_2 \dots a_{2n+1} \begin{vmatrix} x^7 & x^9 & \dots & x^{2n+3} \\ A_4 & A_5 & \dots & A_{n+2} \\ \dots & \dots & \dots & \dots \\ A_{n+1} & A_{n+2} & \dots & A_{2n-2} \end{vmatrix}
$$

$$= (-1)^n a_1^{n-1} (a_2\ a_3)^{n-2} \dots (a_{2n-4}\ a_{2n-3})^1$$

$$\begin{vmatrix} -x^4 q_{2n+1} & {}_{2n}\alpha_2 & 0 & 0 \\ x^2 q_{2n+1} & {}_{2n}\alpha_4 & {}_{2n}\alpha_2 & 0 \\ -q_{2n+1} & {}_{2n}\alpha_6 & {}_{2n}\alpha_4 & {}_{2n}\alpha_2 \\ q_{2n-1} & {}_{2n-2}\alpha_6 & {}_{2n-2}\alpha_4 & {}_{2n}\alpha_2 \end{vmatrix} \quad \dots(24)$$

§ 6. Consider the following equations :—

$$x^{2n+1} - {}_{2n}\alpha_{2n}\, x^{2n-1} + {}_{2n}\alpha_{2n-2}\, x^{2n-3} - \dots + (-1)^n x\, {}_{2n}\alpha_2 = q_{2n+1} \quad (x.)$$

$$y^{2n+1} - {}_{2n}\alpha_{2n}\, y^{2n-1} + {}_{2n}\alpha_{2n-2}\, y^{2n-3} - \dots + (-1)^n y\, {}_{2n}\alpha_2 = Q_{2n+1} \quad (y.)$$

$$A_{n+1} - {}_{2n}\alpha_{2n}\, A_n + {}_{2n}\alpha_{2n-2}\, A_{n-1} - \dots + (-1)^n\, {}_{2n}\alpha_2\, A_1 = 0 \quad \dots (29)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$A_{2n-1} - {}_{2n}\alpha_{2n}\, A_{2n-2} + \quad \dots \quad \dots \quad \dots + (-1)^n\, {}_{2n}\alpha_2\, A^{n-1} = 0.$$

Eliminate the n co-efficients (α's). We obtain :

$$\begin{vmatrix} x^{2n+1} - q_{2n+1} & x^{2n-1} & x^{2n-3} & \dots & x \\ y^{2n+1} - Q_{2n+1} & y^{2n-1} & y^{2n-3} & \dots & y \\ A_{n+1} & A_n & A_{n-1} & \dots & A_1 \\ A_{n+2} & A_{n+1} & A_n & \dots & A_2 \\ \dots & \dots & \dots & \dots & \dots \\ A_{2n-1} & A_{2n-2} & \dots & \dots & A_{n-1} \end{vmatrix} = 0$$

i. e. $$\begin{vmatrix} x & x^3 & \dots & x^{2n+1} \\ y & y^3 & \dots & y^{2n+1} \\ A_1 & A_2 & \dots & A_{n+1} \\ \dots & \dots & \dots & \dots \\ A_{n-1} & A_n & \dots & A_{2n-1} \end{vmatrix} + (-1)^{n-1} q_{2n+1} \begin{vmatrix} y & y^3 & \dots & y^{2n-1} \\ A_1 & A_2 & \dots & A_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_{n-1} & \dots & \dots & A_{2n-2} \end{vmatrix}$$

$$+ (-1)^n Q_{2n+1} \begin{vmatrix} x & x^3 & \dots & x^{2n-1} \\ A_1 & A_2 & \dots & A_n \\ \dots & \dots & \dots & \dots \\ A_{n-1} & A_n & \dots & A_{2n-2} \end{vmatrix} = 0.$$

Hence

$$[x, y, \mathbf{A}_1, \mathbf{A}_{n+1}] = \begin{vmatrix} x & x^3 & \dots & x^{2n+1} \\ y & y^3 & \dots & y^{2n+1} \\ \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_{n+1} \\ \dots & \dots & \dots & \dots \\ \mathbf{A}_{n-1} & \mathbf{A}_n & \dots & \mathbf{A}_{2n-1} \end{vmatrix}$$

$$= [\mathbf{A}_1, \mathbf{A}_{n-1}] \begin{vmatrix} q_{2n-1} & \mathrm{Q}_{2n-1} \\ q_{2n+1} & \mathrm{Q}_{2n+1} \end{vmatrix} \text{[from (7) above]...(30)}$$

From the same equations (29), we have also,

$$1 \div \begin{vmatrix} x^{2n-1} & x^{2n-3} & \dots & x - (-1)^n \dfrac{q_{2n+1}}{2n\alpha_2} \\ y^{2n-1} & y^{2n-3} & \dots & y - (-1)^n \dfrac{\mathrm{Q}_{2n+1}}{2n\alpha_2} \\ \mathbf{A}_n & \mathbf{A}_{n-1} & \dots & \mathbf{A}_1 \\ \dots & \dots & \dots & \dots \\ \mathbf{A}_{2n-3} & \mathbf{A}_{2n-4} & \dots & \mathbf{A}_{n-2} \end{vmatrix}$$

$$= (-1)^n {}_{2n}\alpha_2 \div \begin{vmatrix} x^{2n+1} & x^{2n-1} & .. & x^3 \\ y^{2n+1} & y^{2n-1} & .. & y^3 \\ \mathbf{A}_{n+1} & \mathbf{A}_{n+2} & .. & \mathbf{A}_2 \\ .. & .. & .. & .. \\ \mathbf{A}_{2n-2} & \mathbf{A}_{2n-3} & .. & \mathbf{A}_{n-1} \end{vmatrix}$$

$$\begin{vmatrix} x^3 & x^5 & \dots & x^{2n+1} \\ y^3 & y^5 & \dots & y^{2n+1} \\ \mathbf{A}_2 & \mathbf{A}_3 & \dots & \mathbf{A}_{n+1} \\ \mathbf{A}_3 & \mathbf{A}_4 & \dots & \mathbf{A}_{n+2} \\ \dots & \dots & \dots & \dots \\ \mathbf{A}_{n-1} & \mathbf{A}_n & \dots & \mathbf{A}_{2n-2} \end{vmatrix} = (-1)^n {}_{2n}\alpha_2 \left\{ \begin{vmatrix} x & x^3 & \dots & x^{2n-1} \\ y & y^3 & \dots & y^{2n-1} \\ \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_n \\ \dots & \dots & \dots & \dots \\ \mathbf{A}_{n-2} & \mathbf{A}_{n-1} & \dots & \mathbf{A}_{2n-3} \end{vmatrix} \right.$$

$$+(-1)^{n+1}\frac{q_{2n+1}}{{}_{2n}\alpha_2}\begin{vmatrix} y^3 & \cdots & y^{2n-1}\\ A_2 & \cdots & A_n \\ \cdots & \cdots & \cdots\end{vmatrix} +(-1)^{n+2}\frac{Q_{2n+1}}{{}_{2n}\alpha_2}\begin{vmatrix} x^3 & \cdots & x^{2n-1}\\ A_2 & \cdots & A_n\\ \cdots & \cdots & \cdots\end{vmatrix}\Bigg|\Bigg\}$$

$$= (-1)^n \Big\{ {}_{2n}\alpha_2 \, [A_1, \; A_{n-2}] \begin{vmatrix} q_{2n-3} & Q_{2n-3}\\ q_{2n-1} & Q_{2n-1}\end{vmatrix}$$

$$- q_{2n+1}\begin{vmatrix} Q_{2n-1} & {}_{2n-2}\alpha_2 \\ -Q_{2n-3} & {}_{2n-4}\alpha_2\end{vmatrix}[A_1, A_{n-2}]$$

$$+ Q_{2n+1}\begin{vmatrix} q_{2n-1} & {}_{2n-2}\alpha_2\\ -q_{2n-3} & {}_{2n-4}\alpha_2\end{vmatrix}[A_1, A_{n-2}]$$

$$= (-1)^n \, [A_1, A_{n-2}]\begin{vmatrix} q_{2n-3} & q_{2n-1} & q_{2n+1}\\ Q_{2n-1} & Q_{2n-1} & Q_{2u+1}\\ {}_{2n-4}\alpha_2 & {}_{2n-2}\alpha_2 & {}_{2n}\alpha_2\end{vmatrix} \qquad \dots \qquad (31)$$

Other very general results of this type will be dealt with in a future paper.

THE DIFFERENTIAL CALCULUS IN ANCIENT INDIA*

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When I was recently reading that ever interesting book, Mrs. Manning's 'Ancient and Mediæval India,' I came across the fact that one of my ancestors did know the Differential Calculus in the 12th century! It was an agreeable surprise; I was transported with joy. Here it is:—

"Mr. Spottiswoode has explained the Hindu astronomical methods of calculating eclipses, for those who wish to comprehend their nature, and estimate their real value......The respect and interest, which the persevering labour and intelligence of ancient Hindus excited in Mr. Spottiswoode, were further evidenced in 1859, when Professor Wilson asked for his opinion on a short article by Bapu Deva Sastri, Professor of Mathematics and Astronomy at the Government College of Benares, the object of which was to show that Bhaskaracharya, who lived at Ougein (Ujjayini) in the twelfth century, 'was fully acquainted with the principle of the Differential Calculus,' that being the most important discovery of the 17th century in Europe."

* * * * *

Mr. Spottiswoode answered Professor Wilson's request referred to above in his letter dated the 5th May 1859 printed *infra*; and in publishing it in the Journal of the Royal Asiatic Society, Professor Wilson observed in his note that as the fact that an Indian astronomer knew the Differential Calculus in the 12th century would have been a remarkable circumstance in the history of Astronomical Science, it was obviously a matter of more than ordinary interest to have the accuracy of Professor Bapu Deva's statement carefully tested. He therefore applied to his colleague, Mr. William Spottiswoode, who was well-known as a mathematician, for his opinion. He did not doubt that the answer with which he was favoured would be thought by the Society worthy of being communicated to the public through the Journal. He added that it showed that Bapu Deva's statement was not correct to its whole extent, yet it did full justice to Bhaskaracharya's penetration and science, and acknowledged that his calculations bore a very remarkable analogy to the corresponding process in modern Mathematical Astronomy.

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* Extracts from two articles which appeared in the *Indian Review* of November, 1920, and October 1921.

Through the courtesy of Mr. McPherson, Librarian of the Madras Literary Society,and Mr. Sivarama Ayyar, Assistant Librarian of the Madras University Library, I have procured a copy of Professor Bapu Deva Sastri's original article, dated the 4th May 1858, which runs as follows :—

"It appears to be generally believed that the principle of the Differential Calculus was unknown to the ancient Hindu mathematicians. Allow me to correct this impression by the following statement regarding what Bhaskaracharya has written on this subject.

Bhaskaracharya says that "the difference between the longitudes of a planet found at any time on a certain day and at the same time on the following day is called its rough motion during that interval of time and that its *Tatkalika* motion is its exact motion."

The *Tatkalika* or instantaneous motion of a planet is the motion which it would have in a day, had its velocity at any given instant of time remained uniform. This is clear from the meaning of the term *Tatkalika* and it is plain enough to those who are acquainted with the principles of the Differential Calculus that this *Tatkalika* motion can be no other than the differential of the longitude of a planet. This *Tatkalika* motion is determined by Bhaskaracharya in the following manner :—

"Suppose x, x' = the mean longitudes of a planet on two successive days :

y, y' = the mean anomalies ;

u, u' = the true longitudes ;

and a = eccentricity or the sine of the greatest equation of the orbit.

Then, $x' - x$ = the mean motion of the planet, $y' - y$ = the motion of the mean anomaly and $u' - u$ = the true motion of the planet. "Now, according to Bhaskaracharya, the equation of the orbit on the first day = $\frac{a \sin y}{\text{Rad}}$, and that on the next day = $\frac{a \sin y'}{\text{Rad}}$;

$$\therefore \quad u = x - \frac{a \sin y}{\text{Rad}}, \quad \ldots \quad \ldots \ (1)$$

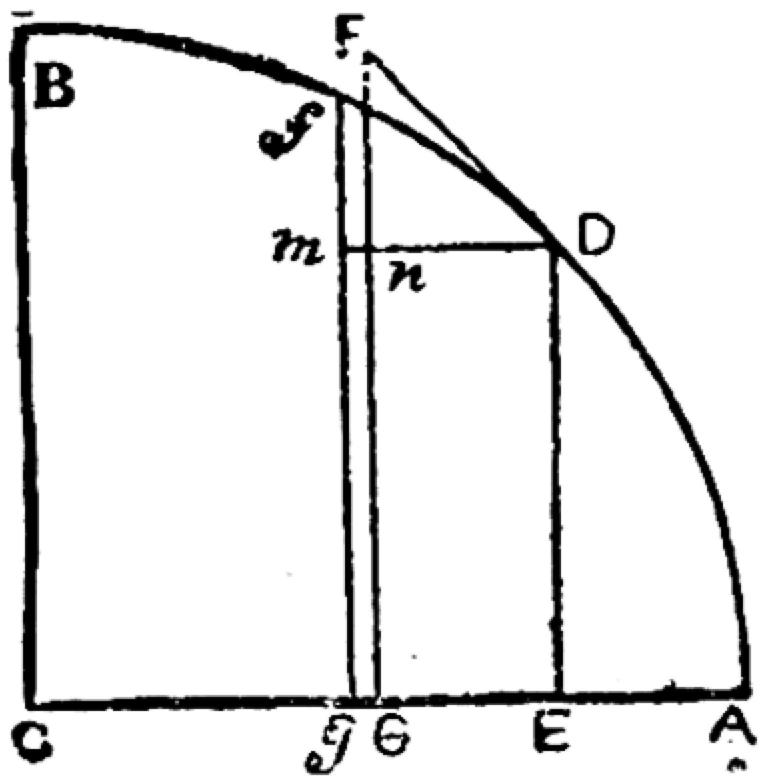
and

$$u' = x' - \frac{a \sin y'}{\text{Rad}} ;$$

$$\therefore \quad u' - u = x' - x - \frac{a (\sin y' - \sin y)}{\text{Rad}} . \quad \ldots \ (2)$$

Now, in order to know the instantaneous value of $u' - u$, it is necessary first to know the instantaneous value of the *Bhogya-khanda* or the difference between two successive sines given in *Tables of Sines*. Thus, suppose the sines of the arcs 0, A, 2A, 3A, &c., are given in the *Tables of Sines*, then $\sin A - \sin 0$, $\sin 2A - \sin A$, $\sin 3A - \sin 2A$, &c., are the *Bhogya-khandas*.

"These are not equal to each other but gradually decrease, and consequently while the increase of the arc is uniform, the increment of the sine varies"—on account of the deflection of the arc. Hence the difference between any two successive sines is not the *Tatkalika Bhogya-khanda* ; but if the arc instead of being deflected be increased in the direction of the tangent, then the increase which would take place in the sine is the *Tatkalika Bhogya-khanda*, *i.e.*, the instantaneous motion of the sine.



Thus, in the accompanying diagram, suppose the arc $Df = A$, then, $\sin Af - \sin AD = fg - DE = fm$, the *Bhogya-khanda* of the sine DE ; but this is not the *Tatkalika Bhogya-khanda* of that sine. If the arc AD instead of being deflected towards f, be increased in the direction of the tangent, so that $DF = Df = A$; then $FG - DE = Fn$, which would be the *Tatkalika Bhogya-khanda* of the sine DE, *i.e.*, the instantaneous motion of that sine."

Bhaskaracharya has determined that "the *Tatkalika Bhogya-khanda* varies as the cosine of the arc, *i.e.*, when arc$=0$, its cosine equals the radius and A =the *Tatkalika Bhogya-khanda*. And, as the arc increases, the cosine and the *Bhogya-khanda* decrease. Hence, if y be any given arc, the *Tatkalika Bhogya-khanda* answering to it will be found by the following proportion :—

As R (or the cosine when arc$=0$) : *Tatkalika Bhogya-khanda* $(=A)$
:: Cosine y : *Tatkalika Bhogya-khanda* of $\sin y$.

$$\therefore \textit{Tatkalikà Bhogya-khanda} \text{ of } \sin y = \frac{A . \cos y}{R} .\text{"}$$

The reason of the above proportion can be easily understood from the two similar triangles DCE and DFn in the above diagram.

"After having thus determined the *Tatkalika Bhogya-khanda*, the instantaneous value of($\sin y' - \sin y$) is found by the following proportion:—

$A : \frac{A \cos y}{R} :: y'-y : \frac{(y'-y)\cos y}{R}$ (= the instantaneous value of $\sin y' - \sin y$.)

By substituting the instantaneous value of ($\sin y' - \sin y$) in the equation (2), the instantaneous value of $u'-u$, the true motion of the planet will be found; that is,

$$u'-u = (x'-x) - \frac{a}{R} \cdot \frac{(y'-y)\cos y}{R} \cdot \ldots \qquad (3)$$

This is the instantaneous motion of the planet."

This is the way in which Bhaskaracharya determined the instantaneous motion of the sun and the moon.

Equation (3) is just the differential of equation (1). For,

$$d\,(u) = d\,(x - \frac{a \sin y}{R});$$

$d(u) = d($

or

$$du = dx - \frac{a}{R} \cdot \frac{\cos y}{R}\,dy,$$

which is similar to equation (3).

Now, the term *Tatkalika* applied by Bhaskaracharya to the velocity of a planet, and his method of determining it, correspond exactly to the differential of the longitude of a planet, and the way for finding it. Hence it is plain that Bhaskaracharya was fully acquainted with the principle of the Differential Calculus. The subject, however, was only incidentally and briefly treated by him; and his followers, not comprehending it fully, have hitherto neglected it entirely."

Here is Mr. Spottiswoode's reply to Professor Wilson :—

"12, James Street, Buckingham Gate,
London, May 5, 1859.

My Dear Mr. Wilson,

I have read Bapu Deva Shastri's letter on Bhaskaracharya's mode of determining the instantaneous motion of a planet, with great interest and think that we are much indebted to him for calling our attention to so important an element in the old Indian methods of calculation. It still, however, seems to me, that he has overstated the case, in saying that "Bhaskaracharya was fully acquainted with the principle of the Differential Calculus." He has undoubtedly conceived the idea of comparing the successive positions of a planet in its path, and of regarding its motion as constant during the interval, and he may be said to have had some rudimentary notion of representing the arc of a curve by means of auxiliary straight lines. But, on the other hand, in the method here given, he makes no allusion to one of the most essential features of the Differential Calculus, *viz.*, the infinitesimal magnitude of the intervals of time and space therein employed. Nor, indeed, is anything specifically said about the fact that the method is an approximative one.

Nevertheless, with these reservations, it must be admitted, that the penetration shown by Bhaskara, in his analysis, is in the highest degree remarkable; that the formula which he establishes (equation 3) and his method of establishing it, bear more than a mere resemblance—they bear a strong analogy—to the corresponding process in modern mathematical astronomy; and that the majority of scientific persons will learn with surprise the existence of such a method in the writings of so distant a period and so remote a region.

With many thanks for communicating the paper to me,

I remain,
Very sincerely yours,
(Signed) W. SPOTTISWOODE.

H. H. Wilson, Esq.

P. S.—I may perhaps add, that if—

x, x' be the mean longitudes,
y, y' be the mean anomalies,
u, u' be the true anomalies

of a planet on two successive days, and a the eccentricity or sine of the greatest equation of the orbit; then $(u' - u)$, or the true motion of the planet

$$= x' - x - a\,(\sin y' - \sin y).$$

And Bhaskara's method consists in showing, that the "Instantaneous" value of $(\sin y' - \sin y)$ (or the value which it would have if the velocity of the planet had remained uniform during the day) is $(y' - y)\cos y$. His formula therefore becomes

$$u' - u = x' - x - (y' - y)\,a\cos y.$$

And the corresponding formula in modern analysis is

$$du = d\,(x - a\sin y) = dx - a\cos y\,dy.$$

(Initialled) W. S."

I have quoted above the opinion of Mr. Spottiswoode who, acknowledging Bhaskaracharya's 'penetration and science', stated that his (Bhaskara's) mode of calculations in connection with the differential calculus bore a remarkable *analogy* to the corresponding process in modern Mathematical Astronomy.

In the second volume of "A History of Hindu Chemistry" by Sir Praphulla Chandra Ray, D. Sc., Ph. D., I find the inclusion of a valuable contribution on "the Mechanical, Physical and Chemical Theories of the Ancient Hindus" by Brajendra Nath Seal, M.A., PH. D. In it 'Bhaskara's claim to be regarded as a precursor of Newton in the discovery of the principle of the Differential Calculus' is discussed.

In Professor Seal's book entitled "Positive Sciences of the Ancient Hindus", the contribution referred to above is re-published with the following pregnant remark as an *additional* foot-note appended to the portion relating to Bhaskara's differential calculus with reference to Mr. Spottiswoode's observation that he (Bhaskara) makes no allusion to one of the most essential features of the differential calculus, *viz.*, the infinitesimal magnitude of the intervals of time and space therein employed :—

"It is an error to suppose that infinitesimals are indispensable to the calculus, but I am here speaking of the earlier calculi and not of the modern developments which have made the calculus independent of infinitesimals. My claim on behalf of Bhaskara is limited to the historically earlier imperfect form of the calculus."

I beg leave to quote the following extract bearing on the subject from Professor Seal's contribution, which is at once interesting, instructive and informative; I have transliterated the Sanskrit quotations therein and given a free English translation of the same wherever necessary.

"MEASURES OF TIME AND SPACE. SIZE OF ATOMS."

The *Siddhanta-Siromani* gives the following measures of time :— 30 Kshanas = 1 day, 2 Ghatikas = 1 Kshana, 30 Kalas = 1 Ghatika, 30 Kashthas = 1 Kala, 18 Nimeshas = 1 Kashtha, 30 Tatparas = 1 Nimesha, and 100 Trutis = 1 Tatpara.

This makes a Truti of time equal to 1/33750 of a second, which is nearly the measure of the Paramanu of time, as given in the Vishnupurana (*vide* Bhaskara's *Siddhanta-Siromani—Kalamanadhyaya.*)

The above measures were in use among the astronomers, but the physicists computed according to the following table given both in Udayana's *Kiranavali* and Sridhara's *Nyayakandali* :—30 Muhurtas = 1 day (24 hours), 30 Kalas = 1 Muhurta, 30 Kashthas = 1 Kala, 18 Nimeshas = 1 Kashtha, 2 Lavas = 1 Nimesha, 2 Kshanas = 1 Lava

"Kshanadvayam lavah prokto Nimeshastu lavadvayam
Ashtadasa nimeshastu Kashtha Trimsat tu tah kala
Trimsat Kalah Muhurtah syat Trimsat ratriahani cha te."

—(Udayana, *Kiranavali*).

This makes 1 Kshana of the Nyaya-Vaiseshika equal to $\frac{2}{45}$ of a second. The Nyaya assumes that the unit of physical change (or the time occupied by any single antecedent step in a causal series before the succeeding step is ushered in) is equal to a Kshana (or $\frac{2}{45}$ of a second.) The astronomers were familiar with far smaller measures of time. The astronomical Truti of time measures about the thirty-four-thousandth part of a second. This is of special value in determining the exact character of Bhaskara's claim to be regarded as the precursor of Newton in the discovery of the principle of the Differential Calculus, as well as in the application to astronomical problems and computations. This claim, as I proceed to show, is absolutely established; it is indeed far stronger than Archimedes's to the conception of a rudimentary process of Integration. Bhaskara in computing the "instantaneous motion" (Tatkalikigati) of a planet compares its *successive* positions, and regards its motion as constant during the interval (which of course cannot be greater than a Truti of time, though it may be indefinitely less). This Tatkalika motion is no other than the differential of the planet's longitude, and Bapudeva Sastri claims that both the conception of the instantaneous motion and *the method of determining it* plainly show that Bhaskara was acquainted with the principle of the Differential Calculus. On the data before him, Mr. Spottiswoode remarks that Bapudeva Sastri "overstates the case." Bhaskara "makes no allusion to one of the most essential features of the Differential Calculus, *viz.*, the infinitesimal magnitude of the intervals of time and space therein employed. Nor indeed is anything specifically said about the fact that the method is an approximate one." "With all these reservations," Mr. Spottiswoode continues, "it must be admitted that the formula he establishes and the method of establishing it bear a strong analogy to the corresponding process in modern mathematical astronomy" (*viz.*, the determination of the differential of the planet's longitude,—by no means the first step in transcendental analysis or in its application to astronomy). And Mr. Spottiswoode concludes by stating that mathematicians in Europe will be surprised to learn of the existence of such a process in the age of Bhaskara (*Circa* 1150 A. D.—born 1114 A. D.). Mr. Spottiswoode's second objection that Bhaskara does not specifically state that the method of the calculus is only approximative cannot be taken seriously. The conception of limit and the computation of errors came later in the history of the Calculuses of Fluxions and Infinitesimals. For the rest, Bhaskara introduces his computation expressly as a "correction" of Brahmagupta's rough simplification. The first objection (*viz.*, that Bhaskara makes no allusion to the infinitesimal magnitude of the intervals of space and time employed) would be more to the point, if it were well founded. But it is not, and Mr. Spottiswoode's error was due to the insufficiency of the data supplied to him. As a matter of fact, even Bhaskara's unit, the Truti of time (or Paramanu), is exceedingly small as the very name implies, being about one thirty-four-thousandth of a second of time. And in the passage in which Bhaskara describes the process, he distinguishes between Sthula-gati and Sukshma-gati (velocity roughly measured, and measured accurately, *i. e.*, by reference to indefinitely small quantities, for Sukshma, as we have seen, has always a reference to the Anu, the indefinitely small). Indeed he expressly mentions that the Sthula-gati takes only Sthula-kala (finite time) into consideration, and that the determination of the Tatkaliki-gati (Sukshma-gati) must have reference to the moment (Pratikshanam) which is an indefinitely small quantity of time being of course smaller than his unit, the Truti.

Cf. "Iyam kila sthoola gatih; atha sookshma tatkaliki katthyathe... yatha asannah tithyantah tatha tatkalikya gatya tithi sadhanam kartum yujyate: tatha sameepa chalanam cha: yatha tu duratarah tithyantah durachalanam va chandrasya tatha adyaya sthulaya kartum yujyate sthulakalatvat. Yatah chandragatih mahatvat pratikshanam sama na bhavati atah tadartham ayam visesho abhihitah——this then is the approximate velocity; next is described the exact and instantaneous (velocity)...... When the ending moment of a *tithi* is near, then the proper course to work out the tithi is by means of the instantaneous velocity: so also in the case of (the moon's) motion when near. When, however, the ending moment of a tithi is far off or moon's motion distant, then on account of the longer interval of time the proper method is to employ the first (mentioned) approximate (velocity). The moon's motion, on account of its rapidity, is not uniform every instant, and this is said to be a peculiarity." Nothing can be clearer than this conception of "momentary" motion.—Bhaskara, *Siddhanta Siromani*, *Ganitadhyaya*, *Gatisphutiprakarana*: *cf.* also "Pratikshanam sana sama mahatvatah,—on account of greatness it is not the same every moment," *ibid.*, *cf.* also *Goladhyaya*, *Tatkalikikarana Vasana prakarana*, where Bhaskara points out that the mode of computing adopted by the Acharya (Brahmagupta) is a rough simplification. The computation of relative motion and the idea of resolved components of motion were of course familiar to the astronomers.

Cf. "Evam satyacharyena laghavartham ishta—ghateesambandhinyo gatikalah arke prakshipatah—this being the case, the calculation of velocity at a desired time was based on the sun for the sake of simplicity by the Acharya (Brahmagupta)," *ibid.* "I may add *en passant* that Bhaskara's formula for the computation of a Table of Sines also implies his use of the principle of the Differential Calculus."

SHORT NOTES.

Conics associated with a given point and the Triangle of Reference.

Salmon (*Conic Sections*: Notes on 'Pascal's Theorem') states that the two points g defined by

$$\left\{\begin{matrix} ab,\ de,\ cf \\ cd,\ fa,\ be \\ ef,\ bc,\ ad \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} ab,\ cd,\ ef \\ de,\ fa,\ bc \\ cf,\ be,\ ad \end{matrix}\right\}$$

are conjugate with respect to the circum-conic of the hexagon *abcdef*, a result due to Steiner. Professor Wilkinson has given a proof (*J. I. M. S.*, Vol. X. pp. 349—359) based on statical considerations of force-diagrams and funicular polygons. The following is a proof based on elementary analytical methods.

Consider a triangle ABC and a homothetic triangle A'B'C', having P as the homothetic centre.

Referred to the triangle ABC, let the equations to B'C', C'A', A'B' be, in areals,

$$\begin{aligned} x &= \lambda\, l\,(x + y + z) \\ y &= \lambda\, m\,(x + y + z) \\ z &= \lambda\, n\,(x + y + z). \end{aligned}$$

The co-ordinates of P are l, m, n; and by varying λ, we get a series of homothetic triangles A'B'C', having P as the homothetic centre.

If A'B', A'C' meet BC in α', α; B'C', B'A', meet CA in β', β; and C'A', C'B' meet AB in γ', γ, the points $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ lie on a conic.
The co-ordinates of α, α' are

$$[0,\ \lambda m,\ 1 - \lambda m] \text{ and } [0,\ 1 - \lambda n,\ \lambda n].$$

The co-ordinates of the points can be similarly written and the equation to the conic $\alpha\alpha'\beta\beta'\gamma\gamma'$ is found to be

$$\lambda \Sigma mn\,(1 - \lambda l)\, x^2 - \Sigma\,[\, l\, \{2\lambda^2 mn - \lambda\,(m + n) + 1\}\ yz] = 0,$$

or $$lyz + mzx + nxy = (x + y + z)\, \Sigma\, \lambda mn\,(1 - \lambda l)\, x = 0. \quad \ldots \quad (1)$$

The co-ordinates of the centre of the conic are proportional to

$$[l\,(m + n - l) + \lambda\, l\,(mn + nl - m^2 - n^2)]\text{; etc.} \qquad \ldots (1{\cdot}1)$$

The equation of the polar of P (l, m, n) with respect to the conic is

$$\{2 - \lambda\,(l + m + n)\} \left[\frac{x}{l} + \frac{y}{m} + \frac{z}{n}\right] = \lambda\,\{3 - 2\lambda\,(l + m + n)\}\ [x + y + z] \ldots (1{\cdot}2)$$

The equation to the Pascal line $\begin{pmatrix} \alpha\alpha', & \beta\beta', & \gamma\gamma' \\ \beta\gamma' & \gamma\alpha', & \alpha\beta' \end{pmatrix}$ is

$$\left(\frac{x}{l} + \frac{y}{m} + \frac{z}{n}\right) = \lambda\,(x + y + z). \quad \ldots \qquad (1{\cdot}3)$$

From (1·2) and (1·3) *it follows that the polar of P with respect to the conic* $\alpha\alpha'\beta\beta'\gamma\gamma'$ *is parallel to the Pascal* $\begin{pmatrix} \alpha\alpha', & \beta\beta', & \gamma\gamma' \\ \beta\gamma', & \gamma\alpha', & \alpha\beta' \end{pmatrix}$.

The above result gives a proof of Steiner's Theorem referred to at the beginning of this note. The point P (l, m, n) is a g-point and its corresponding g-point is the intersection of the Pascals

$$\begin{pmatrix} \alpha\alpha', & \beta\beta', & \gamma\gamma' \\ \beta\gamma', & \gamma\alpha', & \alpha\beta' \end{pmatrix} \text{ and } \begin{pmatrix} \alpha\alpha', & \beta\beta', & \gamma\gamma' \\ \beta'\gamma, & \gamma'\alpha, & \alpha'\beta \end{pmatrix}.$$

This latter is the line at infinity. Since the Polar of P with respect to the conic is parallel to the first Pascal, it follows that the point P and its corresponding g-point which is at infinity are conjugate with respect to the conic, whence the general theorem is deduced by projection.

The following results are noticed :—

(i) For variable values of λ, the conic $\alpha\alpha'\ \beta\beta'\ \gamma\gamma'$ touches the conic

$$\left(\frac{x}{l}\right)^{\frac{1}{2}} + \left(\frac{y}{m}\right)^{\frac{1}{2}} + \left(\frac{z}{n}\right)^{\frac{1}{2}} = 0$$

inscribed in the triangle ABC ... ... (2)

(ii) The centre of the conic lies on the line

$$\Sigma\, mn\,(m - n)\, x = 0 \quad \ldots \quad \ldots \quad (2{\cdot}1)$$

passing through P (l, m, n).

(iii) The polar of P with respect to the conic moves parallel to the line

$\frac{x}{l} + \frac{y}{m} + \frac{z}{n} = 0$ and the Pascal line $\begin{pmatrix} \alpha\alpha', & \beta\beta', & \gamma\gamma' \\ \beta\gamma', & \gamma\alpha', & \alpha\beta' \end{pmatrix}$

is also parallel to the same line. ... ... (2·2)

Particular Cases.

Designating the conic $\alpha\alpha'\beta\beta'\gamma\gamma'$ determined by a particular value of λ as the P_λ-conic, when $\lambda = 0$, the triangle A′B′C′ coincides with the triangle ABC and the equation of the P_0-conic is

$$\frac{l}{x} + \frac{m}{y} + \frac{n}{z} = 0\,; \quad \ldots \quad \ldots \quad (3)$$

and the polar of P with respect to this conic is

$$\frac{x}{l} + \frac{y}{m} + \frac{z}{n} = 0. \quad \ldots \qquad \ldots \quad (3{\cdot}1)$$

When $\lambda = \dfrac{1}{x + y + z}$, the points A′,B′,C′ coincide with P. Calling the Pλ-conic in this case as the P-conic, the following results may be deduced :

The centre of the P-conic bisects the distance between the point P and the centre of P_0*-conic.* ... ... (4)

The Pascal of the P-hexagon, *viz.*, $\frac{x}{l} + \frac{y}{m} + \frac{z}{n} = 0$ is the polar of P with respect to the P-conic. ... ... (4·1)

When $\frac{l}{a^2} = \frac{m}{b^2} = \frac{n}{c^2}$, the P$\lambda$-conics become the well-known Tucker's circles, from whose properties we can deduce corresponding ones for Pλ-conics by orthogonal projection.

The radius Rλ of the Tucker circle Tλ, is given by the relation

$$R_\lambda^2 = R^2 \left[1 - \lambda \textstyle\sum l + \lambda^2 \sum mn\right],$$

where R is the circum-radius of ABC ... ... (5)

Since the triangle $\alpha\beta\gamma$ is similar to ABC,

$$\frac{R_\lambda^2}{R^2} = \frac{\Delta\alpha\beta\gamma}{\Delta ABC} = 1 - \lambda \textstyle\sum l + \lambda^2 \sum mn.$$

For any particular position of A′ B′ C′, the equation of the circle A′B′C′ is

$$\sum \frac{a^2}{x - \lambda a^2 (x + y + x)} = 0,$$

which may also be written in the form

$$\textstyle\sum a^2 yz = \lambda \sum x \left[2 \sum b^2c^2x - 3\lambda a^2b^2c^2 \sum x\right].$$

F. H. V. GULASEKHARAM.

Pencils of Conics.

1. In the pencil formed by the four rays

$$L = k_r\,L' \quad (r = 1, 2, 3, 4),$$

we know that a cross-ratio is equal to $\dfrac{(k_1 - k_2)(k_3 - k_4)}{(k_1 - k_4)(k_3 - k_2)}$. ... (1)

Conics passing through four fixed points are said to form a *pencil*; and if $S = 0$, $S' = 0$ denote two of these, $S = kS'$ represents any conic of the pencil. Let us take four conics (S_1, S_2, S_3, S_4) of this pencil, corresponding to the parameters k_r $(r = 1, 2, 3, 4)$. The object of this note is to obtain a geometrical meaning of (1) in relation to these conics.

2. *Definitions.*

Let any transversal cut the conics S_1, S_2, S_3, S_4 in the points (P_1, P_1') (P_2, P_2'), (P_3, P_3') (P_4, P_4'). Take four points, one from each pair, and form a cross-ratio: *e.g.* $(P_1\,P_2\,P_3\,P_4)$. Replacing the first and the third (or the second and the fourth) by their mates, we get a cross-ratio $(P_1'P_2P_3'P_4)$ $\{$or $(P_1P_2'P_3P_4')\}$ which may be said to be '*alternate*' to the original cross-ratio. The product of a cross-ratio and its alternate may be called a *super-cross product*. It is evident that the four pairs of points give rise to sixteen super-cross products.

We shall now prove the following :—

3. THEOREM I. *The sixteen super-cross products of the four pairs of points are equal and each is equal to the expression (1).*

Let the transversal cut the conics S, S′ in (A, B), (A′, B′). These six pairs of points (A,B) (A′,B′) (P_1, P_1') (P_2, P_2') ...... belong to an involution (Desargues' Theorem) whose centre is O, say, and whose constant is c.

Taking O as origin and the transversal as x-axis, let us denote the abscissæ of the points (P_r, P_r') by (x_r, x_r') and those of (A,B), (A′,B′) by (α, β), (α', β'), so that

$$\alpha\beta = \alpha'\beta' = x_r\,x_r' = c \qquad \ldots \qquad \ldots \quad (2)$$

If $\quad S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$

and $\quad S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c,$

then α, β are the roots of the equation

$$ax^2 + 2gx + c = 0,$$

and α', β' are the roots of

$$a'x^2 + 2g'x + c = 0,$$

Putting $y = 0$ in $S = k_r S'$, we see that x_r, x_r' are the roots of

$$ax^2 + 2gx + c = k_r (a' x + 2g' x + c)$$

i.e., of
$$a (x - \alpha) (x - \beta) = k_r \, a' (x - \alpha') (x - \beta').$$

Now
$$\frac{a'}{a} (k_1 - k_2) = \frac{(x_1 - \alpha) (x_1 - \beta)}{(x_1 - \alpha') (x_1 - \beta')} - \frac{(x_2 - \alpha) (x_2 - \beta)}{(x_2 - \alpha') (x_2 - \beta')}$$
$$= \frac{(x_1 - x_2) (x_1 x_2 - c) (\alpha + \beta - \alpha' - \beta')}{(x_1 - \alpha') (x_1 - \beta') (x_2 - \alpha') (x_2 - \beta')}$$

by taking account of (2).

$$\therefore \quad \frac{(k_1 - k_2) (k_3 - k_4)}{(k_1 - k_4) (k_3 - k_2)}$$
$$= \frac{(x_1 - x_2) (x_3 - x_4)}{(x_1 - x_4) (x_3 - x_2)} . \frac{(x_1 x_2 - c) (x_3 x_4 - c)}{(x_1 x_4 - c) (x_3 x_2 - c)}.$$

On the right hand side, use

$$c = x_2 x_2' = x_4 x_4'.$$

Thus
$$\frac{(k_1 - k_2) (k_3 - k_4)}{(k_2 - k_4) (k_3 - k_2)}$$
$$= (P_1 \, P_2 P_3 \, P_4 .). \frac{(x_1 - x_2') (x_3 - x_4')}{(x_1 - x_4') (x_3 - x_2')}$$
$$= (P_1 \, P_2 \, P_3 \, P_4) . (P_1 \, P_2' \, P_3 \, P_4').$$

which is one of the super-cross-products.

From this, follows :

THEOREM II. *Any transversal cuts a pencil of four given conics in a range having a constant super-cross-product.*

R. GOPALASWAMI.

SOLUTIONS.

Question 939.

(S. R. **Ranganathan**) :—If ρ, ρ_0 and ρ_α are the radii of curvature at the corresponding points of a curve, its evolute, and its α-evolute respectively, show that as α varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, ρ_α has for its maximum value the expression $\sqrt{\rho_0^2 + \rho^2}$, which it attains when $\tan\alpha = \frac{\rho}{\rho_0}$.

Solution by S. Audinarayanan.

It has been shown in question 854, that for a curve in general

$$\rho_\alpha = \rho\left(\frac{d\rho}{ds}\cos\alpha + \sin\alpha\right).$$

We know that $\rho = \frac{ds}{d\psi}$ and $\rho_0 = \frac{d\rho}{d\psi}$, where ψ is the angle that the tangent at the point makes with the x-axis.

$$\therefore\ \rho_\alpha = \rho\left(\frac{d\rho}{d\psi}\cdot\frac{d\psi}{ds}\cos\alpha + \sin\alpha\right)$$

$$= \rho\left(\frac{\rho_0}{\rho}\cos\alpha + \sin\alpha\right)$$

$$= \rho_0\cos\alpha + \rho\sin\alpha.$$

The value of α for which ρ_α has a turning value is given by

$$\frac{d\rho_\alpha}{d\alpha} = 0$$

i.e., $\quad -\rho_0\sin\alpha + \rho\cos\alpha = 0,$

i.e., $\quad \tan\alpha = \frac{\rho}{\rho_0}.$

For this value of α, $\frac{d^2\rho_\alpha}{d^2\alpha}$ is negative; hence ρ_α is a maximum.

The maximum value is

$$\rho_\alpha = \frac{\rho_0\rho_0}{\sqrt{\rho_0^2+\rho^2}} + \rho\cdot\frac{\rho}{\sqrt{\rho_0^2+\rho^2}}$$

$$= \sqrt{\rho_0^2+\rho^2}.$$

Thus the maximum value is given by the expression $\sqrt{\rho_0^2+\rho^2}$ and the value of α is given by $\tan\alpha = \frac{\rho}{\rho_0}$.

Question 942.

(**Hemraj**) :—Show that if p be prime, $\sum_{1}^{p-1}\left(\frac{1}{n}\right) \div \frac{p^3}{p!}$ is an integer.

Additional solution by A. Mahalingam.

$$\sum_{1}^{p-1}\left(\frac{1}{n}\right) \div \frac{p^8}{p!} = \left(1+\frac{1}{2}+\frac{1}{3}\dots+\frac{1}{p-1}\right) \div \frac{p^3}{p!} \doteq \frac{S_{p-2}}{p^2},$$

where S_r is the sum of all products of the numbers 1, 2, 3, ... $(p-1)$ taken r at a time.

$$(p-1)! = (p-1)(p-2)\dots(p-\overline{p-1})$$

$$(p-1)! = p^{p-1} - S_1 p^{p-2} \dots + S_{p-3}\, p^2$$

$$- S_{p-2}\, p \quad + (p-1)!$$

Hence $\quad p^{p-2} - S_1 p^{p-3} \dots + S_{p-3}\, p - S_{p-2} = 0.$

But by Lagrange's Theorem $S_1, S_2 \dots S_{p-3}$ are multiples of p, p being prime.

$$\therefore \quad S_{p-2} = M(p^2)$$

whence $\frac{S_{p-2}}{p^2}$ is an integer.

Question 970.

(**S. R. Ranganathan**) :—Show that $x^n \prod_{k=1}^{\infty}\left(1+\frac{x^{2n}}{k^{2n}}\right)$ is equal to

$$\frac{1}{2^{\frac{n}{2}}\pi^n}\prod_{r=1}^{n-1}(\cosh 2\pi\alpha_r\ x - \cos 2\pi\beta_r\ x)$$

when n is even, and equal to

$$\frac{1}{2^{\frac{1}{2}(n-1)}\pi^n}\sinh\pi x\prod_{1}^{n-1}(\cosh 2\pi\alpha_r\ x - \cos 2\pi\beta_r\ x)$$

when n is odd, $\alpha_r \equiv \sin\frac{r\pi}{2n}$ and $\beta_r \equiv \cos\frac{r\pi}{2n}$, r being an odd integer.

Remarks by Hemroj.

This question occurs in Hobson's *Trigonometry*: Ch. XVII, Ex. 33 and may be solved on lines similar to the solution of Q. 921 published on p. 444, Vol. X of this Journal.

Question 984.

(S. R. **Ranganathan**) :—A prism has for its right-section a quadrilateral ABCD with $B=75^\circ$, $C=105^\circ$ and $D=90^\circ$ and $\sqrt{3}\ AB=\sqrt{2}\ BC$. Show that a ray of light incident on the face AB at the angle $\sin^{-1}(\mu/2)$ and emerging out after once being totally reflected would undergo a deviation of 90°, μ being the co-efficient of refraction of the substance of the prism.

Solution by S. Audinarayanan.

Since the ray is incident at an angle $\sin^{-1}\frac{\mu}{2}$ and since the co-efficient of refraction is μ, the angle of refraction is that whose sine is $\frac{1}{2}$, *i.e.* 30°.

Suppose the ray after refraction at P meets the face BC. The angle of incidence here will be 45°. It is given that the ray is reflected once totally; let this be reflected and meet the face AD. The angle of incidence will be 30° and so the angle of emergence is $\sin^{-1}\frac{\mu}{2}$. Hence the total deviation from the original path is given by

$$\left(\sin^{-1}\frac{\mu}{2}-30^\circ\right)+90^\circ-\left(\sin^{-1}\frac{\mu}{2}-30^\circ\right)$$

which is 90°.

There are two other cases. Suppose the ray is reflected in BC and DC in succession. Then the angle of incidence on AD is 30° and the ray emerges out at an angle equal to $\sin^{-1}\frac{\mu}{2}$. Here the total deviation is

$$\left(\sin^{-1}\frac{\mu}{2}-30^\circ\right)+90^\circ+60^\circ+\sin^{-1}\frac{\mu}{2}-30^\circ=90^\circ+2\sin^{-1}\frac{\mu}{2}.$$

Again we have to consider also the case when the ray is incident to the right of the normal.

The ray will meet the face AD at an angle of incidence 60° and so will be totally reflected. This ray will meet the face DC at an incident angle 30° and so will emerge out at an angle $\sin^{-1}\frac{\mu}{2}$ to the normal. In this case the deviation will be $2\sin^{-1}\frac{\mu}{2}$.

Since the angle CAB is readily seen to be 60°, further cases do not arise.

Question 1020.

(**Anantharaman**) :—Construct a magic square of 81 cells with the following numbers at the places assigned :—

1st Row : 5th cell 25.
2nd ,, : 4th cell 79 ; 5th cell 29 ; 6th cell 27.
3rd ,, : 3rd cell 43 ; 5th cell 81 ; 7th cell 5.
4th ,, : 2nd cell 7 ; 5th cell 13 ; 8th cell 19.
5th ,, : (in full) 61, 47, 9, 67, 41, 15, 73, 35, 21.
6th ,, : 2nd cell 63 ; 5th cell 69 ; 8th cell 75.
7th ,, : 3rd cell 77 ; 5th cell 1 ; 7th cell 39.
8th ,, : 4th cell 55 ; 5th cell 53 ; 6th cell 3.
9th ,, : 5th cell 57.

Solution by S. Audinarayanan.

The following square satisfies the conditions :—

45	10	65	30	*25*	80	51	4	59
64	44	12	*79*	*29*	*27*	58	50	6
11	66	*43*	26	*81*	28	*5*	60	49
48	*7*	62	42	*13*	68	36	*19*	74
61	*47*	*9*	*67*	*41*	*15*	*73*	*35*	*21*
8	*63*	46	14	*69*	40	20	*75*	34
33	22	*77*	54	*1*	56	*39*	16	71
76	32	24	*55*	*53*	*3*	70	38	18
23	78	31	2	*57*	52	17	72	37

Question 1088.

(**A. A. Krishnaswami Aiyangar**) :—Prove that the orthopole of one of the asymptotes of a rectangular hyperbola with reference to any inscribed triangle is the foot of the perpendicular from the orthocentre of the triangle on the other asymptote.

Solution by K. Satyanarayana.

Let A, B, C, the vertices of any inscribed triangle be respectively

$\left(c\,t_1, \frac{c}{t_1}\right) \left(c\,t_2, \frac{c}{t_2}\right) \left(c\,t_3, \frac{c}{t_3}\right)$ the asymptotes being taken as axes.

The equation of the perpendicular on BC from the foot of the perpendicular from A on $y = 0$ is $y = (x - ct_1)\, t_2 t_3$, which cuts $x = 0$ at the point $(0, -ct_1\, t_2 t_3)$ which should be the orthopole of $y = 0$, with respect to the triangle ABC.

The equations of the perpendiculars from A and B respectively on BC, CA are

$$t_1 y - x\, t_1 t_2 t_3 = c - c\, t_1{}^2 t_2 t_3$$
$$t_2 y - x\, t_1 t_2 t_3 = c - c\, t_1 t_2{}^2 t_3$$

giving the 'y' co-ordinate of the orthocentre as $-c\, t_1 t_2 t_3$.

Hence the result.

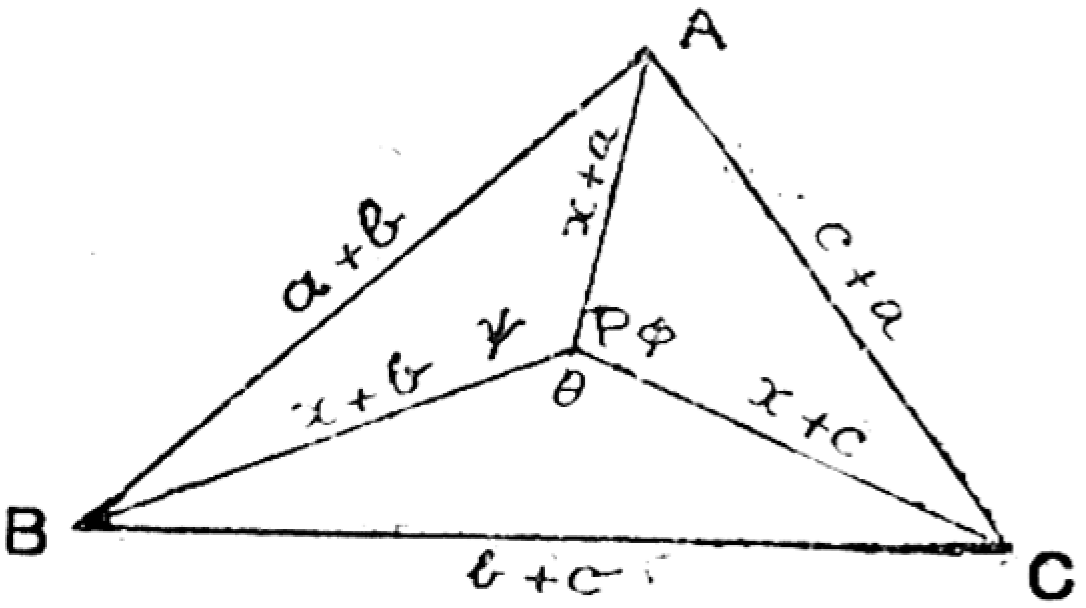
Question 1091.

(ENQUIRER) :—If a, b, c be the radii of three circles which touch one another externally, and r_1, r_2 the radii of the two circles drawn to touch these three, show that

$$\frac{1}{r_1} \sim \frac{1}{r_2} = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Additional Solution by G. S. Mahajani.

The question occurs in Loney's *Plane Trigonometry*, Part I, page 249, Ex. 30. The following is an elementary solution :—



Let the circle whose centre is P and radius x touch the circles whose centres are A, B, C and radii a, b, c respectively.

Then obviously $\quad \text{PA} = x + a$; &c.

Now, $\quad \theta + \phi + \psi = 2\pi$

$\therefore\ \Sigma \sin^2 \dfrac{\theta}{2} = 2 + 2\, \Pi \cos \dfrac{\theta}{2}$ ... [Loney : *Trigonometry*, Pt. 1. Page 135, Ex. 3.]

$$\text{Now,}\quad \sin^2\frac{\theta}{2} = \frac{(\overline{x+b+c} - \overline{x+b})(\overline{x+b+c} - \overline{x+c})}{(x+b)(x+c)}$$
$$= \frac{bc}{(x+b)(x+c)}$$

Similarly $\quad \cos^2 \frac{\theta}{2} = \frac{(x+b+c)\,x}{(x+b)\,(x+c)}.$

Hence substituting, we get

$$\Sigma \frac{bc}{(x+b)\,(x+c)} - 2 = 2 \ \Pi \left\{ \frac{x\,(x+b+c)}{(x+b)\,(x+c)} \right\}^{\frac{1}{2}}$$

i.e. $\quad \Sigma\, bc\,(x+a) - 2\,\Pi\,(x+a) = 2\ \{ x^3\,\Pi\,(x+b+c) \}^{\frac{1}{2}}$

i. e. $\quad \{ 2x^3 + 2\,\Sigma a.\,x^2 + \Sigma\, bc.\,x - abc \}^2 = 4x^3\,\Pi\,(x+b+c).$

It is easily seen that all terms in x^3 and higher powers cancel out, and that the above is a quadratic equation.

If x_1 and x_2 be the two roots,

$$\frac{1}{x_1} + \frac{1}{x_2} = - \frac{\text{co-efficient } x}{\text{constant term}}$$

$$= - \frac{-2\,abc\,\Sigma\, bc}{(abc)^2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$

Having regard to the nature of contact of the two circles, this means

$$\frac{1}{r_1} \sim \frac{1}{r_2} = \frac{1}{x_1} + \frac{1}{x_2} = 2\ \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right),$$

where r_1, r_2 are the numerical values of the radii of the circles.

Question 1095.

(R. S. NARASIMHAM) :—Seven thieves A, B, C ... secure a sum of rupees, but are obliged to conceal it without counting it. A returns alone, divides the sum into 7 parts, finds that there are 6 rupees over, takes the 6 rupees and one-seventh part and departs. B returns alone and does the same with the diminished sum. He divides it into 7 parts, finds that there are 5 rupees over, takes the 5 rupees and one-seventh part and departs. Each does the same, the successive remainders being 6, 5, 4, 3, 2, 1, 0. Finally they all come together and divide the remainder which is a multiple of 7, into 7 equal parts and each takes one part. Find the least number of rupees stolen and the amount that each gets.

Additional Solution and remarks by A. A. Krishnaswami Aiyangar.

The solution of the above question by its author is based on the principle of continued fractions, but it is possible to generalise the problem and solve it as under :

Instead of 7 in the question, let there be n thieves and the successive remainders after each division be $n-1$, $n-2$, ... 2, 1, 0.

Let x be the total number of rupees and $p_1, p_2, \dots$ the successive quotients after each division.

Then we get the following system of equations :—

$$\begin{aligned}
x &= np_1 + (n-1),\\
(n-1)\,p_1 &= np_2 + (n-2),\\
\dots \quad & \dots \quad \dots \quad \dots\\
(n-1)\,p_r &= np_{r+1} + (n-r-1),\\
(n-1)\,p_{r+1} &= np_{r+2} + (n-r-2),\\
(n-1)\,p_{r+2} &= np_{r+3} + (n-r-3),\\
\dots \quad & \dots \quad \dots \quad \dots\\
(n-1)\,p_{n-2} &= np_{n-1} + 1, \qquad \dots\\
(n-1)\,p_{n-1} &= np_n,\\
(n-1)\,p_n &= np_{n+1}.
\end{aligned}$$

We have

$$\left.\begin{aligned}
p_1 &= kp_2 + \frac{n-2}{n-1}\\
p_2 &= kp_3 + \frac{n-3}{n-1}\\
\dots \quad & \dots \quad \dots\\
p_{n-1} &= kp_n
\end{aligned}\right\}, \text{ where } k = \frac{n}{n-1}.$$

$$\therefore \quad p_1 = k^{n-1}p_n + \frac{[1.\,k^{n-3} + 2.\,k^{n-4} + \dots (n-3)\,k + n - 2]}{n-1}.$$

Now if $s = 1.k^{n-3} + 2.k^{n-4} + \dots + (n-2)$,

$$ks = k^{n-2} + 2.k^{n-3} + 3.\,k^{n-4} + \dots (n-2)\,k,$$

$$\therefore \; s\,(k-1) = (k^{n-2} + k^{n-3} + \dots 1) - (n-1)$$

$$= \frac{k^{n-1} - 1}{k-1} - (n-1)$$

$$\therefore \; \frac{s}{n-1} = (k^{n-1} - 2)\,(n-1)$$

Hence $p_1 = k^{n-1} p_n + (n-1)(k^{n-1} - 2)$,

i.e., $(p_1+2n-2)=k^{n-1}(p_n+n-1) = \left(\frac{n}{n-1}\right)^{n-1}(p_n+n-1)$

$$\therefore \quad \left.\begin{aligned} p_1 + 2n - 2 &= n^{n-1} . t, \\ \text{and} \quad p_n + n - 1 &= (n-1)^{n-1} . t \end{aligned}\right\}$$

Substituting in

$$x = np_1 + (n-1),$$

we get

$$\begin{aligned} x &= n^n . t - n(2n-2) + n - 1 \\ &= n^n . t - (2n^2 - 3n + 1) \\ &= n^n . t - (2n-1)(n-1). \end{aligned}$$

Also, since $p_n = kp_{n+1}$,

$$n(p_{n+1}) = (n-1)^n t - (n-1)^2.$$

Hence

$$\begin{aligned} t(-1)^n - 1 &= 0 \ (\text{mod. } n) \\ &= nk, \text{ say.} \end{aligned}$$

Thus finally x and p_{n+1} can be expressed in terms of an arbitrary integer k.

Question 1107.

(Charles Saldanha) :—If x is any odd integer not divisible by 3, prove that the integral part of $\{4^x - (2 + \sqrt{2})^x\}$ is a multiple of 112.

Additional Solution by Salva.

Since $(2 - \sqrt{2})^x$ is a proper fraction, the question means that

$$[4^x - (2 + \sqrt{2})^x - (2 - \sqrt{2})^x]$$

is a multiple of 112.

To prove this, put $a = 2 + \sqrt{2}$, $b = 2 - \sqrt{2}$, then $a + b = 4$.

$$\therefore \quad 4^x - (2 + \sqrt{2})^x - (2 - \sqrt{2})^x = \{(a+b)^x - a^x - b^x\},$$

which is divisible by $ab(a+b)$, since it vanishes when $a = 0$, $b = 0$, $a + b = 0$.

Hence, the expression is a multiple of 8. ... ... (1)

Again, put $a = b\omega$, where ω is a cube-root of unity.

$$\begin{aligned} \text{Exp}^{\text{n}}. = \{(1+\omega)^x - \omega^x - 1\}\ b^x &= (-\omega^{2x} - \omega^x - 1).b^x \\ &= 0, \end{aligned}$$

since x is of the form $(6m + t)$.

Hence, the expression is divisible by

$$(a - b\omega)(a - b\omega^2) = (a^2 + b^2 + ab)$$
$$= 2(2^2 + 2) + 2 = 14. \qquad (2)$$

Combining (1) and (2), we have the result stated.

[*A similar method is suggested by A. A. Krishnaswami Iyengar.*]

Question 1123.

(G.S.M.):—Find a formula for the number of regions into which space of three dimensions is divided by n planes.

Remarks by the Author.

I should have liked the solution to be given in the following elegant form :—

(i) A point being indivisible can be said to consist of

$${}^{n}C_0 \text{ parts};$$

(ii) A straight line is divided by n points into

$$({}^{n}C_0 + {}^{n}C_1) \text{ parts};$$

(iii) A plane is divided by n straight lines into

$$({}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2) \text{ parts};$$

(iv) Space (of three dimensions) is divided by n planes into

$$({}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3) \text{ parts};$$

(v) We may generalize and say that "space of r dimensions is divided by n things of $\overline{r-1}$ dimensions into

$${}^{n}C_0 + {}^{n}C_1 + \ldots. + {}^{n}C_r \text{ parts}.$$

The following particular case deserves notice :—

Space is divided by n planes all passing through *a point* (*not a line*) into

$$2({}^{n}C_0 + {}^{n}C_2) \text{ parts}.$$

In other words, the surface of a sphere is divided by n *great* circles into

$$2({}^{n}C_0 + {}^{n}C_2) \quad \text{portions}.$$

Thus the number of spherical triangles formed by three great circles is

$$8 = 2({}^{3}C_0 + {}^{3}C_2).$$

Question 1137.

(Martyn M. Thomas) :—Two particles moving with velocities $1 : k$ are at corresponding points of a curve and its n^{th} negative pedal. Show that the differential equation of the original curve is

$$\frac{dr}{d\theta} = r^{(n-1)/n} \cdot \sqrt{(kr+\alpha)^{2/n} - r^{2/n}},$$

where 'α' is an arbitrary constant.

Each radius vector of the curve $r = b \cos(\theta \sin \alpha)$ is diminished by $b \cos \alpha$. Show that this new curve and its negative pedal can be described by two particles simultaneously with velocities $\sec \alpha : 1$.

Solution by N. Sundaram Aiyar.

If dashes refer to the n^{th} negative pedal, we have with the usual notation

$$\tan\phi = \frac{r d\theta}{dr}; \quad r' = \frac{r}{\sin^n \phi}; \quad \delta s' = k\,\delta s.$$

and $$\cos\phi = \frac{dr}{ds} = \frac{dr'}{ds'} \quad \therefore\ dr' = k\,dr$$

or $$r' = kr + a = r \operatorname{cosec}^n \phi$$

$$= r\left[1 + \left(\frac{dr}{r d\theta}\right)^2\right]^{n/2}$$

$$\therefore \quad (kr + a)^{2/n} - r^{2/n} = r^{2/n-2}\left(\frac{dr}{d\theta}\right)^2,$$

or $$\frac{dr}{d\theta} = r^{(n-1)/n}\sqrt{(kr + \alpha)^{2/n} - r^{2/n}}. \quad \ldots \quad (1)$$

The equation to the given curve is

$$r = b \cos(\theta \sin \alpha) - b \cos \alpha.$$

$$\therefore \quad \frac{dr}{d\theta} = -\sin\alpha \,.\, b \sin(\theta \sin \alpha).$$

$$\therefore \quad \left(\frac{dr}{d\theta}\right)^2 = \sin^2\alpha\,\{b^2 - (r + b\cos\alpha)^2\}$$

$$= (r\cos\alpha - b\sin^2 a)^2 - r^2.$$

This is the same as the differential equation (I) with $a' = -b\sin^2\alpha$, $k = \cos\alpha$, $n = 1$.

Hence the given curve and its first negative pedal can be described by two particles simultaneously with velocities in the ratio $(1 : \cos a)$ or $\sec a : 1$.

QUESTIONS FOR SOLUTION.

1270. (R. **Vaidyanathaswami**) :—If a quadric has double three-point contact with a twisted cubic, shew that it touches all the osculating planes of the curve.

1271. (K. **Satyanarayana**) :—(i) p being any odd prime, prove that

$$2^{p-1} - 1 \not\equiv 0 \ (\text{mod}\ .\ p^2).$$

Hence or otherwise establish that the numerator of

$$\left[1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots + (-1)^{\frac{p+1}{2}} \frac{1}{\frac{p-1}{2}}\right] \not\equiv 0 \ (\text{mod}\ .\ p);$$

(ii) p being a prime > 3, prove that

$$\left\{(p-1)! - (-1)^{\frac{p-1}{2}} \left[2^{p-1}\left(\frac{p-1}{2}\right)!\right]^2\right\} \equiv 0 \ (\text{mod}\ .\ p^3).$$

1272. (A. C. L. **Wilkinson**) :—Solve the differential equation

$$(xy - 1)\, x \frac{dy}{dx} + 2(y - x^2) = 0.$$

1273. (S. **Audinarayanan**) :—Fill up the vacant cells in the following square with integers in G. P. such that the continued product of the numbers along each row and column and along the diagonals is the same.

1			
			2

1274. (A. A. **Krishnaswami Iyengar**) :—Prove the following approximate construction for the rectification of a circle :—

Draw AB, CD two perpendicular diameters. Along AO take AK = $\frac{1}{8}$ AO. Bisect CO at E. Join CK and from it cut off CG = CO and through G draw GF parallel to KE to meet OE at F. Draw a circle to pass through G and F so as to touch CD at F. Let this circle cut CG again at H.

Then the circumference of the circle ACBD is

$$6\,OC + 2\,CH.$$

1275. (K. J. Sanjana) :—Prove that the equality $6x^4 + 3x = y^2$ is impossible for integral values of x and y, except in the single case when $x = 1$, $y = 3$.

Is the equality possible for other real rational values of x and y.

1276. (G. V. Telang) :—If x represents the area of a triangle and x_1, x_2 those of the pedal triangles of the positive (or negative) Brocard point and the third Brocard point, show that

$$x\,x_2 = 2\,x_1\,(x - 2x_1).$$

1277. (S. Rajanarayanan) :—ABC is a triangle, (D, E), (F, G) are points in AB, AC, such that AD = BE = AF = CG. If DG, FE meet BC in H, K, prove that BH = CK.

1278. (N. P. Pandya) :—Evaluate $\int_{\alpha}^{\beta} \cos(\theta^2)\,d\theta$, without expanding the cosine function in powers of θ.

1279. (Inquirer : G. V. Gurjar) :—Let A and B be two point charges at points $(-a, 0)$ and $(+a, 0)$; the equation of the lines of force in the plane xy can be found as

$$\frac{dy}{dx} = \frac{y}{x + a\left(\dfrac{PB^3 - PA^3}{PB^3 + PA^3}\right)}, \quad \ldots \quad \ldots \quad (1)$$

where P is the point (x, y).

This equation admits of a solution in the form

$$\frac{x + a}{PA} + \frac{x - a}{PB} = \text{constant.}$$

Explain the method of solution.

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ERRATA.

Page	Line	For	Read
226	3	$\frac{1}{x+y+z}$,	$\frac{1}{l+m+n}$.
	10	$\frac{l}{a^2}=\frac{m}{b^2}=\frac{r}{c^2}$,	$l=a^2,\ m=b^2,\ n=c^2$.
	14	$R^2[1-\lambda\Sigma l+\lambda^2\Sigma mn]$,	$R^2[1-\lambda\Sigma a^2+\lambda^2\Sigma b^2c^2]$.
	17	$1-\lambda\Sigma l+\lambda^2\Sigma mn$,	$1-\lambda\Sigma a^2+\lambda^2\Sigma b^2c^2$.

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